

# SPINOR NORM FOR SKEW-HERMITIAN FORMS OVER QUATERNION ALGEBRAS

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ABSTRACT. We complete all local spinor norm computations for quaternionic skew-hermitian forms over the field  $\mathbb{Q}$  of rational numbers. This can be used to compute the number of classes in a genus of skew-hermitian lattices of rank 2 or larger over a maximal order in a quaternion algebra  $D$  over  $\mathbb{Q}$  in many cases, e.g., when  $D$  ramifies at infinity. Examples are provided.

## 1. INTRODUCTION

Let  $K$  be a number field and let  $D$  be a quaternion algebra over  $K$  with canonical involution  $q \mapsto \bar{q}$ . Let  $V$  be a rank- $n$  free  $D$ -module. Let  $h : V \times V \rightarrow D$  be a *skew-hermitian form*, i.e.,  $h$  is  $D$ -linear in the first variable and satisfies  $h(x, y) = -\overline{h(y, x)}$ . A  $D$ -linear map  $\phi : V \rightarrow V$  preserving  $h$  is called an isometry. We denote by  $\mathcal{U}_K$  (resp.  $\mathcal{U}_K^+$ ) the unitary group of  $h$  (resp. the special unitary group of  $h$ ), i.e., the group of isometries (resp. isometries with trivial reduced norm) of  $h$ . Skew-hermitian forms share many properties of quadratic forms. In fact, if  $D \cong \mathbb{M}_2(K)$ , skew-hermitian forms in a rank- $n$  free  $D$ -module are naturally in correspondence with quadratic forms in the  $2n$ -dimensional  $K$ -vector space  $PV$ , for any idempotent matrix  $P$  of rank 1 in  $D$  [?, §3]. In this case, the unitary group of  $h$  is isomorphic to the orthogonal group of the corresponding quadratic form. On the other hand,  $\mathcal{U}_K = \mathcal{U}_K^+$  when  $D$  is a division algebra [?, §2.6].

As in the quadratic case, the problem of determining if two skew-hermitian lattices in the same space are isometric or not can be approached by the theory of genera and spinor genera [?]. This theory depends on the knowledge of the image, under the spinor norm, of the stabilizer of a given lattice in each local group  $\mathcal{U}_{K_{\mathfrak{p}}}^+$ . Full computations of this image exist for symmetric integral bilinear forms. Non-dyadic cases can be found in [?] and dyadic cases in [?]. For this reason we assume, from now on, that  $D$  is a quaternion division algebra. For skew-hermitian forms, non-dyadic places have been completely studied by Böge in [?]. The dyadic case was studied by Arenas-Carmona in [?] and [?], not covering all the cases, which we complete here when  $K_{\mathfrak{p}} = \mathbb{Q}_2$ . From now on  $k = K_{\mathfrak{p}}$  denotes a dyadic local field of characteristic 0.

We denote by  $|\cdot| : D \rightarrow \mathbb{R}_{\geq 0}$  and  $|\cdot|_k : k \rightarrow \mathbb{R}_{\geq 0}$  the absolute values on  $D$  and  $k$  respectively, and assume  $|q| = |Nq|_k$ , where  $N$  is the reduced norm, for any  $q \in D$ . We use  $\nu$  for the surjective valuation  $\nu : D^* \rightarrow \mathbb{Z}$ . Let  $\mathcal{O}_D = \nu^{-1}(\mathbb{Z}_{\geq 0}) \cup \{0\}$  be the unique maximal order in  $D$  [?, §2]. A skew-hermitian lattice or  $\mathcal{O}_D$ -lattice in  $V$ , is a lattice  $\Lambda$  in  $V$  such that  $\mathcal{O}_D\Lambda = \Lambda$ . Any skew-hermitian lattice  $\Lambda$  has a decomposition of the type

$$(1) \quad \Lambda = \Lambda_1 \perp \cdots \perp \Lambda_t,$$

where each indecomposable lattice  $\Lambda_r$  has rank 1 or 2, and the scales satisfy  $\mathfrak{s}(\Lambda_{r+1}) \subset \mathfrak{s}(\Lambda_r)$  [?, §5]. If some  $\Lambda_m$  in the decomposition of  $\Lambda$  has rank 1, then  $\Lambda_m = \mathcal{O}_D s_m$  and  $h(s_m, s_m) = a_m$ . We usually write  $\Lambda_m = \langle a_m \rangle = \mathcal{O}_D s_m$  in this case. A statement like  $\Lambda = \langle a_1 \rangle \perp \cdots \perp \langle a_t \rangle = \mathcal{O}_D s_1 \perp \cdots \perp \mathcal{O}_D s_t$  must be interpreted similarly. Define  $A \subset k^*/k^{*2}$  by  $A = \{N(a_m)k^{*2} | \Lambda_m = \langle a_m \rangle, 1 \leq m \leq t\}$ . Following [?], we define the spinor image  $H(\Lambda) \subseteq k^*$  by the relation  $H(\Lambda)/k^{*2} = \theta(\mathcal{U}_k^+(\Lambda))$ , where  $\mathcal{U}_k^+(\Lambda)$  is the stabilizer of  $\Lambda$

in  $\mathcal{U}_k^+$ , and  $\theta : \mathcal{U}_k^+ \rightarrow k^*/k^{*2}$  denotes the spinor norm. If  $\Lambda = \langle a_1 \rangle \perp \cdots \perp \langle a_n \rangle$ , we let  $\mu(\Lambda) = \min \{ \nu(a_{i+1}) - \nu(a_i) \mid 1 \leq i < n \} \in \mathbb{Z}_{\geq 0}$ . The lattices  $\Lambda$  for which the set  $H(\Lambda)$  remains unknown to date are:

**Case I:**  $\Lambda = \langle a_1 \rangle \perp \cdots \perp \langle a_n \rangle$ , where  $A = \{-uk^{*2}\}$ , for a unit  $u \in \mathcal{O}_k^*$  of non-minimal quadratic defect [?, §63], and  $0 < \mu(\Lambda) \leq \nu(16)$ .

**Case II:**  $\Lambda = \langle a_1 \rangle \perp \cdots \perp \langle a_n \rangle$ , where  $A = \{\pi k^{*2}\}$ , for a prime  $\pi$  in  $k$ , and  $\nu(4) < \mu(\Lambda) \leq \nu(16)$ .

**Theorem 1.** *Table 1 contains all local spinor images when the base field is  $\mathbb{Q}_2$  :*

$s$	$ A $	$A$	$\mu$	$H(\Lambda)$	Reference
—	$> 1$	—	—	$\mathbb{Q}_2^*$	Prop. 2.1 or [?, Table 2]
0	1	$-\Delta\mathbb{Q}_2^{*2}$	—	$\mathbb{Z}_2^*\mathbb{Q}_2^{*2}$	[?, Table 2]
0	1	$-u\mathbb{Q}_2^{*2}$	$0 \leq \mu \leq \nu(4)$	$\mathbb{Q}_2^*$	Prop. 4.1 and [?, Table 1]
0	1	$-u\mathbb{Q}_2^{*2}$	$\mu \geq \nu(8)$	$N(\mathbb{Q}_2(a_m)^*)$	Prop. 4.3 and [?, Table 2]
0	1	$\pi\mathbb{Q}_2^{*2}$	$0 \leq \mu \leq \nu(16)$	$\mathbb{Q}_2^*$	Prop. 4.4 and [?, Tables 1-2]
0	1	$\pi\mathbb{Q}_2^{*2}$	$\mu \geq \nu(32)$	$N(\mathbb{Q}_2(a_m)^*)$	[?, Table 2]
$\neq 0$	—	—	—	$\mathbb{Q}_2^*$	[?, Theorem 2]

TABLE 1. Spinor images for arbitrary lattices over  $\mathbb{Q}_2$ .

Here,  $s$  denotes the number of indecomposable components of rank 2 in the decomposition (1) of  $\Lambda$ , and  $\Delta \in \mathcal{O}_k^*$  is a unit of minimal quadratic defect [?, §63]. Furthermore,  $\pi$ ,  $u$ ,  $A$  and  $\mu = \mu(\Lambda)$  are as above. A dash means “irrelevant”.

Our (computer assisted) proof of Theorem 1 goes as follows: We use Theorem 2 below to reduce the computation of  $H(\Lambda)$  to low rank  $\Lambda$ . In our case, this means rank 2 or 3. Then we use Theorems 3 and 4 for constructing an algorithm for binary lattices over unramified local dyadic fields and we apply it to  $k = \mathbb{Q}_2$ . Then we patch the proof in the remaining “rank 3” case.

Before we state the critical theorems, we recall a few facts about simple rotations in skew-hermitian spaces<sup>1</sup>, see [?, §6] for details. Let  $(V, h)$  be a skew-hermitian  $D$ -space. If  $s \in V$  and  $\sigma \in D^*$  satisfy  $\sigma - \bar{\sigma} = h(s, s)$ , the map  $(s; \sigma) \in \mathcal{U}_K$  defined by  $(s; \sigma)(x) = x - h(x, s)\sigma^{-1}s$  is called a simple rotation with axis  $s$ . Its spinor norm [?] is  $\theta[(s; \sigma)] = N(\sigma)k^{*2}$ , where  $N : D^* \rightarrow k^*$  is the reduced norm. The set of simple rotations span the group  $\mathcal{U}_k^+$ . One way to produce simple rotations, that we use heavily in the sequel, is next lemma:

**Simple Rotation Generating Lemma (SRGL)** [?, Lemma 6.3]: Let  $t, u \in V$  be such that  $h(u, u) = h(t, t) = a$ . Let  $s = t - u$  and  $\sigma = h(t, s)$ . Then  $(s; \sigma)$  is a well-defined simple rotation satisfying  $(s; \sigma)(t) = u$ . Furthermore, if  $u = rt + t_0$ , where  $t_0 \in t^\perp$ , we have the identities  $\sigma = a(1 - \bar{r})$ ,  $h(t_0, t_0) = a - ra\bar{r}$ , and  $\sigma - \bar{\sigma} = h(s, s)$ .

Let  $\Lambda = \langle a_1 \rangle \perp \cdots \perp \langle a_n \rangle$ , with  $\langle a_i \rangle = \mathcal{O}_D s_i$  as in (1). Assume also  $|2a_m| \geq |a_l|$  for  $m < l$ . Then the first author proved in [?, Lemma 6.7] that the unitary group  $\mathcal{U}_k^+(\Lambda)$  of the lattice is generated by  $\mathcal{A}(\Lambda) \cup \mathcal{B}(\Lambda)$ , where

- (1)  $\mathcal{A}(\Lambda)$  is the set of simple rotations with axis  $s_m$ , for some  $m = 1, \dots, n$ .

<sup>1</sup>Some authors call these elements reflections. We prefer the name simple rotation since  $(s; \sigma)$  acts on the 2-dimensional subspace  $k[\sigma]s$  by  $v \mapsto uv$ , where  $u = \bar{\sigma}\sigma^{-1}$  is an element of norm 1.

- (2)  $\mathcal{B}(\Lambda)$  is the set of simple rotations of the form  $(s; \sigma)$ , where  $s = s_m - t$  for some  $t = rs_m + s_0$  with  $s_0 \in \mathcal{O}_{Ds_{m+1}} \perp \cdots \perp \mathcal{O}_{Ds_n}$ , and  $1 - r \notin (2i)$ .

In particular, the elements of  $\mathcal{B}(\Lambda)$  satisfy relations in **SRGL**. Note that  $\mathcal{A}(\Lambda)$  and  $\mathcal{B}(\Lambda)$  depend on the splitting (1).

**Theorem 2.** *Let  $\Lambda = \langle a_1 \rangle \perp \cdots \perp \langle a_n \rangle$ , with  $\langle a_i \rangle = \mathcal{O}_{Ds_i}$  be a skew-hermitian lattice and let  $\mu = \mu(\Lambda)$  be as above. Assume  $\mu > \nu(4)$  and  $N(a_2), \dots, N(a_n) \in N(a_1)k^{*2}$ . Let  $(s; \sigma) \in \mathcal{B}(\Lambda)$ , i.e.,  $s = (1 - r)s_m - s_0$ , where  $s_0 = \lambda_{m+1}s_{m+1} + \cdots + \lambda_n s_n$ ,  $\sigma = a_m(1 - \bar{r})$  and  $|1 - \bar{r}| \geq |2|$ . If  $|\lambda_{m+t}| \geq |\lambda_{m+t+1}|$ , for some  $t \in \{1, \dots, n - m\}$  and for all  $l \in \{1, \dots, n - m - t\}$ , then there exists  $\Lambda' = \langle b_1 \rangle \perp \cdots \perp \langle b_{t+1} \rangle \subset \Lambda$  satisfying the following conditions:*

- (1)  $(s; \sigma) \in \mathcal{U}_k^+(\Lambda')$ .
- (2)  $\mu(\Lambda') \geq \mu(\Lambda)$ .
- (3)  $N(b_i) \in N(a_1)k^{*2}$ , for all  $i = 1, \dots, t + 1$ .

We say that an element  $r \in \mathcal{O}_D$  satisfies the  $k$ -star conditions for a lattice  $\Lambda = \langle a_1 \rangle \perp \cdots \perp \langle a_n \rangle$  if  $z = a_1 - ra_1\bar{r}$  satisfies  $NzNa_1 \in k^{*2}$  and  $NzN(\pi^t a_1)^{-1} \in \mathcal{O}_k$ , where  $\nu(\pi^t) = \mu(\Lambda)$ , while the Hilbert symbol  $\left(\frac{N(1-r), -Na_1}{\mathfrak{p}}\right)$  equals  $-1$ .

**Theorem 3.** *Let  $\Lambda = \langle a_1 \rangle \perp \langle a_2 \rangle$  be a skew-hermitian lattice such that  $|2a_1| \geq |a_2|$  and  $N(a_2) \in N(a_1)k^{*2}$ . The following statements are equivalent:*

- (1)  $H(\Lambda) = k^*$ .
- (2) *There exists  $(s; \sigma) \in \mathcal{B}(\Lambda)$  such that  $N\sigma \notin N(k(a_1)^*)$ .*
- (3) *There exists  $r \in \mathcal{O}_D$  satisfying the  $k$ -star conditions for  $\Lambda$ .*

It is known that the (unique) quaternion division  $k$ -algebra  $D$  has a basis  $\{1, i, j, ij\}$ , where  $i^2 = \pi$ ,  $j^2 = \Delta$ ,  $ij = -ji$ . Moreover, if  $\omega = \frac{j+1}{2}$ , then  $\{1, \omega, i, i\omega\}$  is a  $\mathcal{O}_k$ -basis for  $\mathcal{O}_D$ . Let  $e = \nu(2)/2$  be the ramification index of  $k/\mathbb{Q}_2$ , and assume  $\nu(\pi^t) = \mu(\Lambda)$ .

**Theorem 4.** *Let  $\Lambda$  be as in Theorem 3. There exists  $r \in \mathcal{O}_D$  satisfying the  $k$ -star conditions if and only if there exists  $\alpha \in \mathcal{S} \oplus \mathcal{S}\omega \oplus \mathcal{S}i \oplus \mathcal{S}i\omega \subset \mathcal{O}_D$  satisfying them, for one (any) set of representatives  $\mathcal{S}$  of  $\mathcal{O}_k/\pi^u\mathcal{O}_k$ , with  $u = t + 6e$  as above.*

## 2. GENERATORS OF $\mathcal{U}_k^+(\Lambda)$ AND THEIR SPINOR NORM

If  $\Lambda = \langle a_1 \rangle \perp \cdots \perp \langle a_n \rangle$  is a skew-hermitian lattice, then  $[k^* : N(k(a_1)^*)] = 2$  [?, §63] and  $N(k(a_1)^*) \subset H(\Lambda)$  [?, §6]. As a direct consequence of these facts, we have:

**Proposition 2.1.** *Let  $\Lambda = \langle a_1 \rangle \perp \cdots \perp \langle a_n \rangle$  be a skew-hermitian lattice. Then  $H(\Lambda) = N(k(a_1)^*)$  or  $H(\Lambda) = k^*$ . In particular:*

- (1)  $H(\Lambda) = k^*$  if and only if, there exists  $\phi \in \mathcal{C}(\Lambda)$  such that  $\theta(\phi) \notin N(k(a_1)^*)/k^{*2}$ , for one (any) set of generators  $\mathcal{C}(\Lambda)$  for  $\mathcal{U}_k^+(\Lambda)$ .
- (2) *If there exists  $b \in \mathcal{O}_D$  with  $N(b) \notin N(a_1)k^{*2}$  such that  $\Lambda = \langle b \rangle \perp \Lambda'$ , for some lattice  $\Lambda'$ , we have  $H(\Lambda) = k^*$ .*

**Remark 2.1.** In particular, if  $|2a_m| \geq |a_l|$  for  $m < l$ , and  $\mathcal{C}(\Lambda) = \mathcal{A}(\Lambda) \cup \mathcal{B}(\Lambda)$ , we just need to check the property for the elements in  $\mathcal{B}(\Lambda)$ , since simple rotations  $(s_m; \sigma) \in \mathcal{A}(\Lambda)$  have spinor norm  $N(\sigma)k^{*2} \in N(k(a_m)^*)/k^{*2}$ . Our strategy includes to replace  $\mathcal{B}(\Lambda)$  by a smaller set that still generates but it is easier to control (cf. Lemma 4.6).

**Lemma 2.1.** *Let  $\Lambda = \langle a_1 \rangle \perp \cdots \perp \langle a_n \rangle = \mathcal{O}_{Ds_1} \perp \cdots \perp \mathcal{O}_{Ds_n}$  be a skew-hermitian lattice such that  $|2a_m| \geq |a_l|$  for  $m < l$ . Take  $(s; \sigma) \in \mathcal{B}(\Lambda)$ , i.e.,  $s = (1 - r)s_m - s_0$ , where*

$s_0 = \lambda_{m+1}s_{m+1} + \cdots + \lambda_n s_n$ ,  $\sigma = a_m(1 - \bar{r})$ , and  $|1 - r| \geq |2|$ . If any of the following conditions is satisfied:

- (1)  $|1 - r| > |2|$  and  $|\lambda_{m+1}| < 1$ , while  $\mu(\Lambda) \geq \nu(8)$ , and  $k/\mathbb{Q}_2$  is unramified,
- (2)  $|1 - r| = |2|$  and  $|\lambda_{m+1}| \leq |2|$ , while  $\mu(\Lambda) \geq \nu(4\pi)$ ,
- (3)  $|1 - r| > |2|$  or  $|\lambda_{m+1}| < 1$ , while  $\mu(\Lambda) \geq \nu(16)$ ,
- (4)  $m = 1$  and  $|\lambda_2| \leq |4|$ ,

then  $\theta[(s; \sigma)] \in N(k(a_m)^*)/k^{*2}$ .

Proof. It suffices to prove that if  $a = h(s, s)$ , then  $N(a) \in N(a_m)k^{*2}$ , since  $\sigma \in k(a)$ . In fact, we have  $s = (1 - r)s_m - s_0$ , so that  $a = (1 - r)a_m(1 - \bar{r}) + a_0$ , where  $a_0 = h(s_0, s_0)$ . It follows that

$$(2) \quad N(a) = N(a_m)N(1 - r)^2N(1 + (1 - r)^{-1}a_0(1 - \bar{r})^{-1}a_m^{-1}).$$

Now,  $a_0 = \lambda_{m+1}a_{m+1}\overline{\lambda_{m+1}} + \cdots + \lambda_n a_n \overline{\lambda_n}$  and  $|(1 - r)^{-1}a_0(1 - \bar{r})^{-1}a_m^{-1}| = |a_0||a_m^{-1}|/|1 - r|^2 < |4|$  if any of the conditions above is satisfied. This implies that the last norm in (2) is a square.  $\square$

**Revisando este calculo (arriba) llego a que en la condicion 4 se necesita un poco mas ya que  $|a_0 a_m^{-1}| = |\lambda_{m+1} a_{m+1} \overline{\lambda_{m+1}} a_m^{-1} + \cdots + \lambda_n a_n \overline{\lambda_n} a_m^{-1}| \leq \max\{|\lambda_{m+1} a_{m+1} \overline{\lambda_{m+1}} a_m^{-1}|, \dots, |\lambda_n a_n \overline{\lambda_n} a_m^{-1}|\}$  y esta cantidad no depende solo de  $|\lambda_{m+1}|$ . Por ejemplo, podria ocurrir  $|a_{m+1} a_m^{-1}| = |2|$ ,  $|\lambda_{m+1}|^2 = |16|$  y  $|a_{m+2} a_m^{-1}| = |4|$ ,  $|\lambda_{m+2}|^2 = |4| \dots$**

The following result, together with **SRGL**, give us an easy method to construct simple rotations in  $\mathcal{B}(\Lambda)$  as in the introduction.

**Lemma 2.2.** *Let  $r \in \mathcal{O}_D$  be a non-zero quaternion and let  $a_1, a_2 \in \mathcal{O}_D$  be non-zero pure quaternions. There exists  $\lambda \in \mathcal{O}_D$  different from zero such that  $a_1 = ra_1\bar{r} + \lambda a_2\bar{\lambda}$  if and only if  $NzNa_2 \in k^{*2}$  and  $NzNa_2^{-1} \in \mathcal{O}_k$ , where  $z = a_1 - ra_1\bar{r}$ .*

Proof. The equation  $a_1 = ra_1\bar{r} + \lambda a_2\bar{\lambda}$  has a solution  $\lambda \in D^*$  if and only if the binary skew-hermitian form  $h'$  whose Gramm matrix is  $\begin{pmatrix} z & 0 \\ 0 & -a_2 \end{pmatrix}$  is isotropic. Now,  $h'$  is isotropic if and only if  $NzNa_2 = \text{disc}(h') \in k^{*2}$  [?, Ch. 10, §3, Thm. 3.6]. We conclude that there exists  $\lambda \in D^*$  satisfying  $a_1 = ra_1\bar{r} + \lambda a_2\bar{\lambda}$  if and only if  $NzNa_2 \in k^{*2}$ . Finally, we have  $Nz = Na_2N\lambda^2$ , whence  $\lambda \in \mathcal{O}_D$  if and only if  $NzNa_2^{-1} \in \mathcal{O}_k$ .  $\square$

### 3. PROOF OF THEOREMS 2,3 AND 4

Proof of Theorem 2. Set  $\Lambda' = \langle b_1 \rangle \perp \cdots \perp \langle b_{t+1} \rangle = \mathcal{O}_D s_m \perp \cdots \perp \mathcal{O}_D s_{m+t-1} \perp \mathcal{O}_D s'_0$ , where  $s'_0 = \sum_{i=m+t}^n \lambda_i s_i$ . It is clear that  $\Lambda' \subset \Lambda$ . To prove Condition (1) in the theorem, we note that  $s_0 = s'_0 - \sum_{i=m+1}^{m+t-1} \lambda_i s_i \in \Lambda'$ . We compute

- $(s; \sigma)(s_i) = s_i - h(s_i, s)\sigma^{-1}s = s_i + h(s_i, s_0)\sigma^{-1}s$ , for  $m + 1 \leq i \leq m + t - 1$ ,
- $(s; \sigma)(s'_0) = s'_0 - h(s'_0, s)\sigma^{-1}s = s'_0 + h(s'_0, s_0)\sigma^{-1}s$ , and
- $(s; \sigma)(s_m) = rs_m + s_0 \in \Lambda'$ .

Hence,  $(s; \sigma)(\Lambda') \subseteq \Lambda'$  if  $h(s_i, s_0)\sigma^{-1}, h(s'_0, s_0)\sigma^{-1} \in \mathcal{O}_D$ . The latter holds since  $|\sigma| = |a_m(1 - \bar{r})| \geq |2a_m|$  is larger than the height of  $s_0$ . We conclude that  $(s; \sigma) \in \mathcal{U}_k^+(\Lambda')$ . On the other hand, as

$$b_{t+1} = h(s'_0, s'_0) = \sum_{u=m+t}^n \lambda_u a_u \overline{\lambda_u},$$

we have  $|b_{t+1}| = |a_{m+t}||\lambda_{m+t}|^2$ , since  $|\lambda_{m+t}| \geq |\lambda_{m+t+l}|$  when  $1 \leq l \leq n - m - t$  and  $\mu(\Lambda) > \nu(4)$ . From here  $\mu(\Lambda') \geq \mu(\Lambda)$ , proving Condition (2). Finally, to prove the last condition, we consider

$$(3) \quad N(b_{t+1}) = N(\lambda_{m+t})^2 N(a_{m+t}) N \left( 1 + (\lambda_{m+t} a_{m+t} \overline{\lambda_{m+t}})^{-1} \sum_{u=m+t+1}^n \lambda_u a_u \overline{\lambda_u} \right),$$

where  $|(\lambda_{m+t} a_{m+t} \overline{\lambda_{m+t}})^{-1}| = |a_{m+t}|^{-1} |\lambda_{m+t}|^{-2}$ . Since  $|a_{m+t+l}| < |4a_{m+t}|$  and  $|\lambda_{m+t}| \geq |\lambda_{m+t+l}|$  for all  $l \in \{1, \dots, n - m - t\}$ , the last term in (3) is a square, whence  $N(b_{t+1}) \in N(a_{m+t})k^{*2}$  and the proof of the condition (3) is completed.  $\square$

Proof of Theorem 3. The equivalence between (1) and (2) is a direct consequence of Proposition 2.1 and the subsequent remark. To prove that (2) implies (3), let  $(s; \sigma)$  be a simple rotation such that  $\theta[(s; \sigma)] = N(\sigma)k^{*2} \notin N(k(a_1)^*)/k^{*2}$ . As isometry  $(s; \sigma) \in \mathcal{B}(\Lambda)$  satisfies  $a_1 = h(s_1, s_1) = ra_1\bar{r} + \lambda a_2\bar{\lambda}$ , where  $(s; \sigma)(s_1) = rs_1 + \lambda s_2$ . Such an  $r \in \mathcal{O}_D$  satisfies  $\sigma = a_1(1 - \bar{r})$  by **SRGL**. Hence,  $\theta[(s; \sigma)] \notin N(k(a_1)^*)/k^{*2}$  if and only if  $N(1 - r) \notin N(k(a_1)^*)$ , or equivalently  $\left(\frac{N(1-r), -Na_1}{\mathfrak{p}}\right) = -1$ . On the other hand, Lemma 2.2 tells us that  $NzNa_2 \in k^{*2}$  and  $NzNa_2^{-1} \in \mathcal{O}_k$ , where  $z = a_1 - ra_1\bar{r}$ . The result follows since  $Na_2 \in N(a_1)k^{*2}$  and  $\mu = \nu(a_2) - \nu(a_1) = \nu(\pi^t)$ . Conversely, if  $r \in \mathcal{O}_D$  satisfies the  $k$ -star conditions, then Lemma 2.2 and **SRGL** imply the existence of  $\phi \in \mathcal{B}(\Lambda)$  such that  $\theta(\phi) = N(a_1)N(1 - \bar{r})k^{*2}$  and the result follows as before.  $\square$

**Corollary 3.1.** *Let  $\Lambda$  be as in Theorem 3. Let  $t$  be such that  $\mu = \nu(\pi^t)$ . If  $H(\Lambda) = k^*$ , then  $H(\Lambda') = k^*$  for every lattice  $\Lambda' = \langle b_1 \rangle \perp \langle b_2 \rangle$  with  $N(b_1), N(b_2) \in N(a_1)k^{*2}$  and  $\mu(\Lambda') = \nu(\pi^s)$ , for  $e \leq s < t$ . In particular, if  $L = \langle q \rangle \perp \langle \pi^t q \rangle$ , with  $N(q) \in N(a_1)k^*$ , we have  $H(\Lambda) = H(L)$ .*

**Remark 3.1.** Due to Lemma 2.1, in the condition (2) of Theorem 3, is enough to consider simple rotations  $(s; \sigma) \in \mathcal{B}(\Lambda)$  with  $|\lambda| > |4|$ , where  $s = (1 - r)s_1 - \lambda s_2$ . Remember that  $|1 - r| \geq |2|$  for  $(s; \sigma) \in \mathcal{B}(\Lambda)$ .

Proof of Theorem 4. Assume  $r \in \mathcal{O}_D$  satisfies the  $k$ -star conditions. Let  $\alpha \in \mathcal{O}_D$  be a representative of the class of  $r$  modulo  $\pi^u$  as in the statement. Then,  $r = \alpha + \pi^u \beta$ , with  $\beta \in \mathcal{O}_D$  and  $\alpha \in \mathcal{S} \oplus \mathcal{S}\omega \oplus \mathcal{S}i \oplus \mathcal{S}i\omega \subset \mathcal{O}_D$ . As  $1 - r = 1 - \alpha - \pi^u \beta$  we have  $N(1 - r) = N(1 - \alpha)N(1 - (1 - \alpha)^{-1}\pi^u \beta)$ . Now,  $|1 - r| \geq |2|$  implies  $|1 - \alpha| \geq |2|$ . Therefore,  $N(1 - (1 - \alpha)^{-1}\pi^u \beta)$  is a square. Hence,  $\left(\frac{N(1-r), -Na_1}{\mathfrak{p}}\right) = \left(\frac{N(1-\alpha), -Na_1}{\mathfrak{p}}\right)$ . On the other hand, if  $z = a_1 - ra_1\bar{r} = \pi^t \lambda a_1 \bar{\lambda}$  and  $z' = a_1 - \alpha a_1 \bar{\alpha}$ , then  $z = z' - \pi^u \gamma$ , with  $\gamma = \alpha a_1 \bar{\beta} + \beta a_1 \bar{\alpha} + \pi^u \beta a_1 \bar{\beta} \in \mathcal{O}_D$ . Note that  $a_1^{-1} \gamma \in \mathcal{O}_D$ . We have  $|z'| = |z|$ , since  $|z| = |\pi^t \lambda a_1 \bar{\lambda}| > |16\pi^t a_1| = |\pi^{4e+t} a_1| > |\pi^u \gamma|$ , where we are assuming  $|\lambda| > |4|$  (see Remark 3.1). Furthermore, we have that  $Nz = Nz'N(1 - z'^{-1}\pi^u \gamma)$  with  $|z'^{-1}\pi^u \gamma| < |\pi^{-(4e+t)} a_1^{-1} \pi^{t+6e} \gamma| = |\pi^{2e} a_1^{-1} \gamma| \leq |4|$ . Hence,  $NzNa_1$  is a square if and only if  $Nz'Na_1$  is a square. Finally, from  $|z'| = |z|$  we obtain  $|Nz/\pi^{2t}N(a_1)|_k \leq 1$  if and only if  $|Nz'/\pi^{2t}N(a_1)|_k \leq 1$ .  $\square$

**Remark 3.2.** The optimal choice for the number  $u$  in Theorem 4 depends on  $|\lambda|$ . For example, since  $z = a_1 - ra_1\bar{r} = \pi^t \lambda a_1 \bar{\lambda}$ , if  $\lambda$  satisfies  $|\lambda| = 1$ , then we would have  $|z'| = |\pi^t a_1|$  and so  $|z'^{-1}\pi^u \gamma| = |\pi^{-t} a_1^{-1} \pi^u \gamma| \leq |\pi^{u-t}| < |4|$  if  $u = t + 2e + 1$ . This holds in some cases when  $k = \mathbb{Q}_2$ .

#### 4. PROOF OF THEOREM 1

If  $k = \mathbb{Q}_2$ , we have  $\mathcal{O}_k = \mathbb{Z}_2$ , and  $\mathcal{O}_k/\pi^u \mathcal{O}_k \cong \mathbb{Z}/2^u \mathbb{Z}$ . By considering Theorems 3 and 4, we are able to construct an algorithm for computing  $H(\Lambda)$ , for all binary  $\mathbb{Z}_2$ -lattices  $\Lambda$ , as follows:

- (1) By Corollary 3.1, we are reduced to compute  $H(L)$  for  $L = \langle q \rangle \perp \langle 2^t q \rangle$ , for  $q^2$  running over representative of all suitable square classes, and a few values of  $t$  for each  $q$ .
- (2) Fix a set of representatives  $\mathcal{S}$  of the finite ring  $\mathbb{Z}_2/\pi^u\mathbb{Z}_2$  : We can choose  $\mathcal{S} = \{0, 1, \dots, 2^u - 1\}$  for  $u$  large enough (see remark 3.2).
- (3) For  $r = a + b\omega + ci + di\omega \in \mathcal{S} \oplus \mathcal{S}\omega \oplus \mathcal{S}i \oplus \mathcal{S}i\omega \subset \mathcal{O}_D$ , check if the  $k$ -star conditions are satisfied. This verification can be done by using the computer algebra system Sage [?].
- (4) Conclude that  $H(\Lambda) = \mathbb{Q}_2^*$  if some  $r$  in the last step satisfies the  $k$ -star conditions. Otherwise,  $H(\Lambda) = N(\mathbb{Q}_2(a_1)^*)$  in virtue of Theorem 4 and Proposition 2.1.

**Remark 4.1.** The algorithm can be extended to any unramified finite extension  $k$  of  $\mathbb{Q}_2$ . The condition  $|2a_1| \geq |a_2|$  in Theorems 3 and 2 is essential. Hence, the algorithm does not work, for  $\mu < \nu(2)$ , if the extension  $k/\mathbb{Q}_2$  ramifies, unless the algorithm returns the value  $k^*$  for  $\mu < \nu(2)$ .

**4.1. Computations using Sage.** In all that follows we assume  $i^2 = 2$ ,  $j^2 = 5$ , and  $ij = -ji$ . Whenever a different uniformizing parameter  $\pi$  makes computations easier we use  $i_\pi = u_\pi i$ , for some unit  $u_\pi \in \mathbb{Q}_2(j)$ , such that  $i_\pi^2 = \pi$ , or equivalently  $N(u_\pi) = \pi/2$ . The following results are obtained by computer search. When the algorithm actually find solutions, we actually list them. Otherwise it is just stated that no solutions were found.

**Lemma 4.1** (Table 2). *For any  $q \in \{j + ij, i + j\}$  and  $t \in \{3, 4\}$ , there exist  $r_1, r_2 \in \mathcal{O}_D$  such that:*

- (1)  $|1 - r_1| = |2|$ ,  $NzNq \in \mathbb{Q}_2^{*2}$  and  $NzN(2^t q)^{-1} \in \mathbb{Z}_2^*$ , where  $z = q - r_1 q \bar{r}_1$ .
- (2)  $|1 - r_2| = |i|$ ,  $NzNq \in \mathbb{Q}_2^{*2}$  and  $NzN(2^t q)^{-1} \in \mathbb{Z}_2^*$ , where  $z = q - r_2 q \bar{r}_2$ .

$q$	$t$	$r_1$	$r_2$
$j + ij$	3	$-1 - 4i - 4i\omega$	$1 - 14\omega - i - 10i\omega$
$j + ij$	4	$-1 - 8i - 8i\omega$	$1 - 6\omega - 13i - 6i\omega$
$i + j$	3	$-1 - 4i\omega$	$1 - 2\omega - i$
$i + j$	4	$-1 - 8i\omega$	$-1 - 6\omega - 3i$

TABLE 2. Proof of Lemma 4.1.

**Lemma 4.2** (Table 3). *Let  $L = \langle q \rangle \perp \langle 4q \rangle$  be a skew-hermitian lattice satisfying the conditions in Theorem 3, for  $q \in \{j + ij, i + j\}$ . Then there exists  $r \in \mathcal{O}_D$  satisfying the  $k$ -star conditions for  $L$ .*

$q$	$r$	$N(1 - r)$	$z$	$NzNq$
$j + ij$	$1 + 2i\omega$	$2 \cdot 2^2$	$4(-1 + 2\omega - 4i - 7i\omega)$	$2^4 \cdot 5^2$
$i + j$	$1 + 2i + 2i\omega$	$-2 \cdot 2^2$	$4(1 - 2\omega + 3i + 3i\omega)$	$2^4(1 + 8 \cdot 20)$

TABLE 3. Proof of Lemma 4.2.

**Lemma 4.3** (Table 4). *Let  $L = \langle i_\pi \rangle \perp \langle 16i_\pi \rangle$  be a skew-hermitian lattice satisfying the hypothesis in Theorem 3, for  $\pi \in \{\pm 2, \pm 10\}$  as above. Then, there exists  $r \in \mathcal{O}_D$  satisfying the  $k$ -star conditions for  $L$ .*

**Lemma 4.4.** *There is no  $r = a + b\omega + ci + di\omega \in \mathbb{Z} \oplus \mathbb{Z}\omega \oplus \mathbb{Z}i \oplus \mathbb{Z}i\omega \subset \mathcal{O}_D$ , with  $0 \leq a, b, d, c < 2^{t+3}$  satisfying the  $k$ -star conditions for  $L = \langle q \rangle \perp \langle 2^t q \rangle$ , if  $t \in \{3, 4\}$  and  $q \in \{j + ij, j + i\}$ .*

$\pi$	$r$	$N(1-r)$	$z$	$NzNi_\pi$
$\pm 2$	$15 + 8\omega$	$5 \cdot 2^2(1 + 8 \cdot 5^{-1} \cdot 7)$	$-592i_\pi + 304i_\pi\omega$	$2^{10}(1 + 8 \cdot 38)$
$\pm 10$	$15 + 8\omega$	$5 \cdot 2^2(1 + 8 \cdot 5^{-1} \cdot 7)$	$-592i_\pi + 304i_\pi\omega$	$2^{10} \cdot 5^2(1 + 8 \cdot 38)$

TABLE 4. Proof of Lemma 4.3.

**4.2. Proof of Theorem 1 in Case I.** Assume  $\Lambda = \langle a_1 \rangle \perp \dots \perp \langle a_n \rangle$ , where  $N(a_m) \in -u\mathbb{Q}_2^{*2}$ , for each  $m = 1, \dots, n$  and  $u \in \mathbb{Z}_2^*$  is a unit of non-minimal quadratic defect independent of  $m$ . As  $\mathbb{Z}_2^*/\mathbb{Z}_2^{*2} = \{\pm 1, \pm 5\}$  and a pure quaternion cannot have reduced norm  $-1$ , we have two options for  $u$ :  $u = -5$  or  $u = -1$ .

In virtue of Corollary 3.1, we consider binary lattices  $\Lambda = \langle q \rangle \perp \langle 2^t q \rangle$ , with  $1 \leq t \leq 4$ , where we can choose any pure quaternion  $q \in \mathcal{O}_D^0$  satisfying  $N(q) \in -u\mathbb{Q}_2^{*2}$ . Here,  $q = q_u$  satisfy  $N(q) \in -u\mathbb{Q}_2^{*2}$ , for  $u$  running over the set  $\{-5, -1\}$  of units of non-minimal quadratic defect. We choose  $q_{-5} = j + ij$  and  $q_{-1} = i + j$ .

**Proposition 4.1.** *Let  $\Lambda = \langle a_1 \rangle \perp \dots \perp \langle a_n \rangle$  be a skew-hermitian lattice such that  $N(a_1), \dots, N(a_n) \in -u\mathbb{Q}_2^{*2}$  and  $0 < \mu(\Lambda) \leq \nu(4)$ . Then  $H(\Lambda) = \mathbb{Q}_2^*$ .*

Proof. We can assume  $n = 2$  and  $\Lambda = \langle q_u \rangle \perp \langle 2^t q_u \rangle$ , with  $q_u \in \{q_{-5}, q_{-1}\} = \{j + ij, i + j\}$  and  $t \in \{1, 2\}$ . In virtue of Corollary 3.1 it suffices to prove the result for  $t = 2$ . Lemma 4.2 tells us that there exists  $r \in \mathcal{O}_D$  satisfying the  $k$ -star conditions. This is equivalent to  $H(\Lambda) = \mathbb{Q}_2^*$  by Theorem 3.  $\square$

To handle the cases where  $\mu = \nu(8)$  or  $\mu = \nu(16)$  we use the following result, which is used to improve the set of generators  $\mathcal{B}(\Lambda)$ . The proof is a routine computation.

**Lemma 4.5.** *If  $r \in \mathcal{O}_D$  satisfies either of the equations*

$$(4) \quad j + ij = r(j + ij)\bar{r} + 2^t \lambda(j + ij)\bar{\lambda}, \quad \text{or} \quad i + j = r(i + j)\bar{r} + 2^t \lambda(i + j)\bar{\lambda},$$

where  $\lambda \in \mathcal{O}_D$ , and  $t \geq 2$ , then  $1 - r \in i\mathcal{O}_D$ .

**Lemma 4.6.** *Let  $\Lambda = \mathcal{O}_D s_1 \perp \mathcal{O}_D s_2 = \langle a_1 \rangle \perp \langle a_2 \rangle$  be a skew-hermitian lattice such that  $N(a_1), N(a_2) \in -u\mathbb{Q}_2^{*2}$  and  $\nu(8) \leq \mu(\Lambda) \leq \nu(16)$ . There exists a lattice  $L$  of rank 2 such that  $H(L) = H(\Lambda)$ , and a suitable splitting of  $L$ , such that  $\mathcal{A}(L) \cup \mathcal{B}_l(L)$  generates  $\mathcal{U}_{\mathbb{Q}_2}^+(L)$ , for  $l = 1, 2$ , where  $\mathcal{B}_1(L) = \{(s; \sigma) \in \mathcal{B}(L) : |1 - r| = |i|\}$  and  $\mathcal{B}_2(L) = \{(s; \sigma) \in \mathcal{B}(L) : |\lambda| = 1\}$ , with  $r$  as in **SRGL** and  $\lambda$  as in Lemma 2.2.*

Proof. By Corollary 3.1 there is a lattice  $L = \mathcal{O}_D s_1 \perp \mathcal{O}_D s_2 = \langle q_u \rangle \perp \langle 2^t q_u \rangle$  as above, with  $u \in \{-5, -1\}$  and  $t \in \{3, 4\}$  satisfying  $H(L) = H(\Lambda)$ . Let  $\phi \in \mathcal{B}(L)$  be such that  $\phi(s_1) = r s_1 + \lambda s_2$ . We have  $|1 - r| \in \{|i|, |2|\}$  in virtue of Lemma 4.5. Hence, to prove that  $\mathcal{B}_1(L)$  satisfies the required property, it suffices to prove that, if  $\phi$  satisfies  $|1 - r| = |2|$ , then there exists  $(s; \sigma) \in \mathcal{B}(L)$  such that  $|1 - r''| = |i|$  and  $|1 - r'| = |i|$ , where  $(s; \sigma)(s_1) = r' s_1 + \lambda' s_2$  and  $(s; \sigma)\phi(s_1) = r'' s_1 + \lambda'' s_2$ . In this case, there exists a second element  $(s'; \sigma') \in \mathcal{B}_1(L)$  defined by  $s' = s_1 - (s; \sigma)\phi(s_1)$ ,  $\sigma' = q(1 - \bar{r}'')$  such that  $(s'; \sigma')(s; \sigma)\phi(s_1) = s_1$ . In fact, by a computation we have

$$(5) \quad 1 - r'' = 1 - r + [rq(1 - \bar{r}') + 2^t \lambda q \bar{\lambda}'](1 - \bar{r}')^{-1} q^{-1} (1 - r'),$$

$$(6) \quad \lambda'' = \lambda + [rq(1 - \bar{r}') + 2^t \lambda q \bar{\lambda}'](1 - \bar{r}')^{-1} q^{-1} \lambda'.$$

Lemma 4.1 implies the existence of an element  $r' \in \mathcal{O}_D$  such that  $|1 - r'| = |i|$ ,  $NzN(2^t q) \in \mathbb{Q}_2^{*2}$ , and  $NzN(2^t q)^{-1} \in \mathbb{Z}_2$ , where  $z = q - r' q \bar{r}'$  and  $t \in \{3, 4\}$ . Hence, by Lemma 2.2, there exists  $\lambda' \in \mathcal{O}_D$ , such that  $q = r' q \bar{r}' + 2^t \lambda' q \bar{\lambda}'$ . Then  $(s; \sigma)$ , where  $s = (1 - r')s_1 - \lambda' s_2$  and  $\sigma = q(1 - \bar{r}')$ , belongs to  $\mathcal{B}_1(L)$  (cf. **SRGL**). On the other hand, as

$$(7) \quad |[rq(1 - \bar{r}') + 2^t \lambda q \bar{\lambda}'](1 - \bar{r}')^{-1} q^{-1} (1 - r')| = |rq(1 - \bar{r}') + 2^t \lambda q \bar{\lambda}'| = |1 - \bar{r}'| = |i|$$

and  $|1 - r| = |2|$ , it follows that  $|1 - r''| = |i|$ . In particular,  $\mathcal{A}(L) \cup \mathcal{B}_1(L)$  generates  $\mathcal{U}_{\mathbb{Q}_2}^+(L)$ .

Now, to prove that  $\mathcal{A}(L) \cup \mathcal{B}_2(L)$  generates  $\mathcal{U}^+(L)$ , by a similar argument as for  $\mathcal{B}_1(L)$ , it suffices to prove that, if  $\phi \in \mathcal{B}(L)$  satisfies  $|\lambda| < 1$ , there exists  $(s; \sigma) \in \mathcal{B}(L)$  such that  $|\lambda'| = 1$  and  $|\lambda''| = 1$ , where  $\lambda, \lambda', \lambda''$  are defined by  $\phi, (s; \sigma)$  and  $(s; \sigma)\phi$  respectively, as before. From the equation (6) we see that  $|\lambda''| = 1$  if  $|\lambda| < 1$  and  $|\lambda'| = 1$ . By Lemma 4.1, there exists  $r' \in \mathcal{O}_D$  such that

$$|1 - r'| = |i| \text{ or } |2|, \quad NzN(2^t q) \in \mathbb{Q}_2^{*2} \quad \text{and} \quad NzN(2^t q)^{-1} \in \mathbb{Z}_2^*,$$

where  $z = q - r'q\bar{r}'$  and  $t \in \{3, 4\}$ . Hence, by Lemma 2.2, there exists  $\lambda' \in \mathcal{O}_D$  such that  $q = r'q\bar{r}' + 2^t \lambda' q \bar{\lambda}'$ . Then if  $s = (1 - r')s_1 - \lambda's_2$  and  $\sigma = q(1 - \bar{r}')$ , then  $(s; \sigma) \in \mathcal{B}(\Lambda)$  (cf. **SRGL**), and  $|\lambda'| = 1$  since  $NzN(2^t q)^{-1} \in \mathbb{Z}_2^*$ . Now, we take  $|1 - r'| = |i|$  if  $|1 - r| = |2|$ , and  $|1 - r'| = |2|$  if  $|1 - r| = |i|$ , so that  $|1 - r'|, |1 - r''| \geq |2|$  by (5). The result follows.  $\square$

**Remark 4.2.** Notice that, for a lattice  $\Lambda$  as in the previous lemma, we can replace  $\mathcal{B}(\Lambda)$  by  $\mathcal{B}_l(\Lambda)$ , for  $l = 1, 2$ , in Theorem 3. Hence, since  $|\lambda| = 1$  for  $(s; \sigma) \in \mathcal{B}_2(\Lambda)$ , we can improve the number  $u$  in Theorem 4 in virtue of Remark 3.2.

**Proposition 4.2.** *Let  $\Lambda = \langle a_1 \rangle \perp \langle a_2 \rangle$  be as in Theorem 3. There exists  $r \in \mathcal{O}_D$  satisfying the  $k$ -star conditions for  $t \in \{3, 4\}$  and  $Na_1 \in -u\mathbb{Q}_2^{*2}$ , with  $u$  a unit of non-minimal quadratic defect, if and only if there exists  $\alpha = a + b\omega + ci + di\omega \in \mathbb{Z} \oplus \mathbb{Z}\omega \oplus \mathbb{Z}i \oplus \mathbb{Z}i\omega \subset \mathcal{O}_D$ , with  $0 \leq a, b, c, d < 2^{t+3}$ , satisfying them.*

Combining this result with Theorem 3, Corollary 3.1 and Lemma 4.4, we obtain.

**Corollary 4.1.** *Let  $\Lambda = \langle a_1 \rangle \perp \langle a_2 \rangle$  be a skew-hermitian lattice such that  $N(a_1), N(a_2) \in -u\mathbb{Q}_2^{*2}$ , where  $u$  is a unit of non-minimal quadratic defect and  $\mu = \nu(a_2) - \nu(a_1)$  satisfies  $\nu(8) \leq \mu \leq \nu(16)$ . Then  $H(\Lambda) = N(\mathbb{Q}_2(a_1)^*)$ .*

We need the following result to handle ternary lattices  $\Lambda$  with  $\mu(\Lambda) = \nu(8)$ . For the sake of generality we state it for an arbitrary dyadic field  $k$ .

**Lemma 4.7.** *If  $|\eta| = |i|$  and  $a_1$  is a pure unit, then  $T(2(\eta a_1 \bar{\eta})^{-1} a_1) \in \pi \mathcal{O}_k$ , where  $T$  is the trace map.*

Proof. Set  $\eta = i\rho$ , for  $\rho \in \mathcal{O}_D^*$ . Note that  $a_1 i \equiv i \bar{a}_1 \pmod{\pi}$ , while  $\rho$  and  $\bar{a}_1$  commute modulo  $i$ . We conclude that  $\eta a_1 \bar{\eta} \equiv -N(\rho) \pi \bar{a}_1 \pmod{\pi i}$ . In other words  $\frac{1}{\pi} \eta a_1 \bar{\eta} = -N(\rho) \bar{a}_1 + \varepsilon$ , where  $\varepsilon \in i\mathcal{O}_D$ , whence  $\pi(\eta a_1 \bar{\eta})^{-1} = -(N(\rho) \bar{a}_1)^{-1} + \delta = \frac{-a_1}{N(\rho a_1)} + \delta$ , for some  $\delta \in i\mathcal{O}_D$ . Hence

$$T(2(\eta a_1 \bar{\eta})^{-1} a_1) \equiv \frac{-4a_1^2}{\pi N(\rho a_1)} + \frac{2}{\pi} T(\delta a_1) \pmod{\pi}$$

and the result follows since  $\delta \in i\mathcal{O}_D$  implies  $T(\delta a_1) \in \pi \mathcal{O}_k$ .  $\square$

**Proposition 4.3.** *Let  $\Lambda = \langle a_1 \rangle \perp \dots \perp \langle a_n \rangle$  be a skew-hermitian lattice such that  $N(a_1), \dots, N(a_n) \in -u\mathbb{Q}_2^{*2}$ , where  $u$  is a unit of non-minimal quadratic defect. If  $\mu = \mu(\Lambda)$  satisfies  $\nu(8) \leq \mu \leq \nu(16)$ , then  $H(\Lambda) = N(\mathbb{Q}_2(a_1)^*)$ .*

**Proof when  $\mu = \nu(16)$ .** In virtue of Lemma 2.1 it suffices to consider rotations  $(s; \sigma) \in \mathcal{B}(\Lambda)$  such that  $|1 - r| = |2|$  and  $|\lambda_2| = 1$ . In this case, Theorem 2 tells us we can set  $n = 2$  in the statement of the proposition. For  $n = 2$ , because of Lemma 4.6, we can replace  $\Lambda$  by a lattice  $L$  such that  $H(L) = H(\Lambda)$  and a set of generators of  $\mathcal{U}_{\mathbb{Q}_2}^+(L)$  is  $\mathcal{A}(L) \cup \mathcal{B}_1(L)$ . It follows that  $H(\Lambda) = N(\mathbb{Q}_2(a_1)^*)$  since rotations in  $\mathcal{B}_1(L)$  have spinor norm belonging to  $N(\mathbb{Q}_2(a_1)^*)$  in virtue of Lemma 2.1.  $\square$

**Aquí arriba debería ser en principio  $|\lambda_{m+1}| = 1$  en vez de  $|\lambda_2|$ . Lo mismo en las dos condiciones de abajo.**

**Proof when  $\mu = \nu(8)$ .** In virtue of Lemma 2.1, any rotation  $(s; \sigma) \in \mathcal{B}(\Lambda)$  satisfy  $\theta[(s; \sigma)] \in N(\mathbb{Q}_2(a_1)^*)/\mathbb{Q}_2^{*2}$  unless one of the following conditions is satisfied:

- (1)  $|1 - r| = |i|$ ,  $|\lambda_2| = 1$ ,
- (2)  $|1 - r| = |2|$ ,  $|\lambda_2| \in \{1, |i|\}$ .

As in the previous case, by Theorem 2, when  $|\lambda_2| = 1$  we are reduced to consider binary lattices and when  $|1 - r| = |2|$ ,  $|\lambda_2| = |i|$  to study either binary lattices or rank 3 lattices with  $|\lambda_3| = 1$ . For rank 2 lattices, Corollary 4.1 tells us that  $H(\Lambda) = N(\mathbb{Q}_2(a_1)^*)$ . We prove that, for rank 3 lattices  $\Lambda$  such that  $(s; \sigma) \in \mathcal{B}(\Lambda)$  satisfies  $|1 - r| = |2|$ ,  $|\lambda_2| = |i|$ ,  $|\lambda_3| = 1$  we also have  $\theta[(s; \sigma)] \in N(\mathbb{Q}_2(a_1)^*)/\mathbb{Q}_2^{*2}$ . In fact, in virtue of [?, Lemma 4.3] we can assume that  $\Lambda = \langle a_1 \rangle \perp \langle 8\epsilon_2 a_1 \rangle \perp \langle 64\epsilon_3 a_1 \rangle$ , with  $\epsilon_2, \epsilon_3 \in \mathbb{Z}_2^*$ . Hence, **SRGL** tells us that  $r, \lambda_2, \lambda_3 \in \mathcal{O}_D$ , with  $|1 - r| \geq |2|$ , define an element  $\phi \in \mathcal{B}(\Lambda)$  as before if and only if they satisfy the relation

$$z = a_1 - r a_1 \bar{r} = 8\lambda_2 \epsilon_2 a_1 \bar{\lambda}_2 + 64\lambda_3 \epsilon_3 a_1 \bar{\lambda}_3.$$

We can rewrite this equation as  $z = 8\lambda_3 w \bar{\lambda}_3$ , where  $w = \epsilon_2 \eta a_1 \bar{\eta} + 8\epsilon_3 a_1$  and  $\eta = \lambda_3^{-1} \lambda_2$ . Remember that, in this case,  $|\lambda_2| = |i|$  and  $|\lambda_3| = 1$ . Hence, by Lemma 2.2, the existence of  $r, \lambda_2, \lambda_3$  satisfying the equation above is equivalent to the existence of  $r, \eta \in \mathcal{O}_D$ , with  $|\eta| = |i|$  such that  $NzN(w) \in \mathbb{Q}_2^{*2}$  and  $NzN(8w)^{-1} \in \mathbb{Z}_2$ . We know that  $|w| = |2|$ , so  $NzN(8w)^{-1} \in \mathbb{Z}_2$  if and only if  $\frac{Nz}{2^8} \in \mathbb{Z}_2$ . On the other hand,  $N(w) = N(\epsilon_2 \eta a_1 \bar{\eta}) N(1 + 8\epsilon(\eta a_1 \bar{\eta})^{-1} a_1)$ , where  $\epsilon = \epsilon_2^{-1} \epsilon_3 \in \mathbb{Z}_2^*$ . Here, as  $|8\epsilon(\eta a_1 \bar{\eta})^{-1} a_1| = |4|$ , we can write  $8\epsilon(\eta a_1 \bar{\eta})^{-1} a_1 = 4\epsilon\xi$  with  $\xi = 2(\eta a_1 \bar{\eta})^{-1} a_1 \in \mathcal{O}_D^*$ . As we have the relation  $N(1 + 4\epsilon\xi) = 1 + 4\epsilon T(\xi) + 16\epsilon^2 N(\xi)$ , it follows that  $N(1 + 4\epsilon\xi) \in \mathbb{Q}_2^{*2} \cup 5\mathbb{Q}_2^{*2}$ , hence,  $N(\epsilon_2 \eta a_1 \bar{\eta} + 8\epsilon_3 a_1) \in N(a_1)\mathbb{Q}_2^{*2} \cup 5N(a_1)\mathbb{Q}_2^{*2}$ , where  $N(w) \in 5Na_1\mathbb{Q}_2^{*2}$  if and only if  $T(\xi) \equiv 1 \pmod{2}$ . The last condition is not satisfied in virtue of Lemma 4.7. Therefore, we are reduced to the following result, that is an analogue of Theorem 3:  $H(\Lambda) = \mathbb{Q}_2^*$  if and only if there exists  $r \in \mathcal{O}_D$ , with  $|1 - r| = |2|$  satisfying the conditions:

$$\left( \frac{N(1 - r), -Na_1}{\mathfrak{p}} \right) = -1, \quad NzNa_1 \in \mathbb{Q}_2^{*2}, \quad |z| = |16|.$$

This are the k-star conditions for  $\langle a_1 \rangle \perp \langle 16a_1 \rangle$ , and Lemma 4.4 implies that there is no  $r \in \mathcal{O}_D$  satisfying them. Hence, we conclude that  $H(\Lambda) = N(\mathbb{Q}_2(a_1)^*)$ .  $\square$

**4.3. Proof of Theorem 1 in Case II.** By the discussion at the beginning of the section, in rank 2 case, we consider lattices  $\Lambda$  of the form  $\langle i_\pi \rangle \perp \langle 2^t i_\pi \rangle$ , where  $\nu(4) < t \leq \nu(16)$ , and for every prime  $\pi$ , we set  $i_\pi \in \mathcal{O}_{k(j)} i$  such that  $i_\pi^2 = \pi$ . Remember that, if we prove that  $H(\Lambda) = \mathbb{Q}_2^*$ , then  $H(\Lambda) = \mathbb{Q}_2^*$  for lattices  $\Lambda$  of arbitrary rank. By Corollary 3.1, we can assume  $t = 4$ . Hence, next result follows from Lemma 4.3 and Theorem 3:

**Proposition 4.4.** *Let  $\Lambda = \langle a_1 \rangle \perp \dots \perp \langle a_n \rangle$  be a skew-hermitian lattice such that  $N(a_1), \dots, N(a_n) \in \pi\mathbb{Q}_2^{*2}$  and  $0 < \mu(\Lambda) \leq \nu(16)$ . Then  $H(\Lambda) = \mathbb{Q}_2^*$ .*

**4.4. Examples.** Let consider the family of lattices  $\Lambda = \langle i \rangle \perp \langle 2^t i \rangle$ , for  $t > 0$ , where  $D = \left( \frac{2,5}{\mathbb{Q}} \right)$  and  $i^2 = 2$ .  $D$  ramifies only at 2 and 5. The lattice  $\Lambda$  is unimodular for  $p \neq 2$ . We have that  $H(\Lambda_p) = \mathbb{Z}_p \mathbb{Q}_p^{*2}$  for  $p \neq 2$ , in virtue of the computations in [?] (for  $p \neq 5$ ) and [?, Theorem 4] (for  $p = 5$ ). Hence, the spinor class field  $\Sigma_\Lambda$  can ramify only at 2 and  $\infty$ , so  $\Sigma_\Lambda \subset \mathbb{Q}(\sqrt{-1}, \sqrt{2})$ . Observe that the algebra  $D$  decomposes at infinity and the quadratic form corresponding to  $\Lambda$  is indefinite. Hence, class and spinor genus of  $\Lambda$  coincide and  $\Sigma_\Lambda \subset \mathbb{R}$ . On the other hand, for  $p = 2$ , Table 1 tell us that  $H(\Lambda_2) = \mathbb{Q}_2^*$  if

$t \leq 4$  and  $H(\Lambda_2) = N(\mathbb{Q}_2(i)^*)$  if  $t > 4$ , whence  $\Sigma_\Lambda$  decomposes at 2 for  $t \leq 4$  and ramifies at 2 for  $t > 4$ . We conclude that  $\Sigma_\Lambda = \mathbb{Q}$  for  $t \leq 4$ , while  $\Sigma_\Lambda = \mathbb{Q}(\sqrt{2})$  for  $t > 4$ . In the first case, Hasse principle holds for  $\Lambda$ . In the second case, the class number of  $\Lambda$  is 2.

Now consider the family of lattices  $\Lambda = \langle i \rangle \perp \langle 2^t i \rangle$ , where  $D = \left( \frac{-1, -1}{\mathbb{Q}} \right)$  and  $i^2 = -1$ .  $D$  ramifies only at 2 and  $\infty$ . As before  $H(\Lambda_p) = \mathbb{Z}_p \mathbb{Q}_p^{*2}$  for  $p \neq 2$ , hence  $\Sigma_\Lambda \subset \mathbb{Q}(\sqrt{-1}, \sqrt{2})$ . Since the form has discriminant 1 it is isotropic and therefore  $\mathcal{U}_K^+$  is non-compact at infinity. In fact, the same holds for every binary lattice over a quaternion algebra ramifying at  $\infty$ , since all pure quaternions in the Hamilton Algebra are congruent. Hence, class and spinor genus of  $\Lambda$  coincide also in this case. On the other hand, the spinor image is  $\mathbb{R}^+$  at infinity and, since  $-1$  is a ramified unit at 2, Table 1 tells us that  $\Sigma_\Lambda = \mathbb{Q}$  for  $t \leq 2$ , while  $\Sigma_\Lambda = \mathbb{Q}(\sqrt{-1})$  for  $t > 2$ .

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