Lecture 1
Introduction to the Theory of Spinor Genera

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The theory of spinor genera deals with the classification of a certain types of object of algebraic or arithmetical interest, like a form or an algebraic structure under some equivalence relation, usually arising from the action of an algebraic group. Rather than giving some general definition, we enumerate a few cases where the theory can be applied. So, if we have a set X and a group G acting on X, we will be worried about the orbits $G.x$ (classes). We will consider the following cases:

1. $X = \{$regular quadratic forms of a given dimension (≥ 3) over K or $\mathcal{O}_K\}$ and $G = GL_n(K)$ or $GL_n(\mathcal{O}_K)$. Equivalentely, in the case of integral forms, $X = \{$regular lattices of a given dimension (≥ 3) on $V\}$ and $G = O^+(V)$ (special orthogonal group of the quadratic space V).

2. $X = \{$skew-hermitian forms of a given dimension over K or over a maximal order of a $K-$quaternion division algebra $D\}$ and $G = GL_n(D)$. Equivalently (for integral forms), $X = \{$skew-hermitian lattices of a given dimension over a maximal order of a $K-$quaternion division algebra$\}$ and $G = U^+(V)$ (special unitary group of the skew-hermitian space V).

3. $X = \{$orders of maximal rank in a central simple algebra (CSA) $A\}$ and $G = Aut(A)$ (the group of automorphisms of A).

It is well known that, in the quadratic case over number fields, the classification is achieved by the beautiful Hasse-Minkowski's local-global Theorem. This result is wonderful because the local classification is"very easy" and "very finite" (locally, a class is completely determined by dimension, discriminant and Hasse symbol). Unfortunately, this is the only case (among the ones listed above) where we have this nice local-global Theorem or Hasse principle[1].

[1] For the skew-hermitian case over fields, see Kneser's *Lectures On Galois Cohomology of Classical Groups*, Tata Institute of Fundamental Research, Bombay, 1969. For maximal orders it can be shown that all maximal orders are locally conjugated but, in general, there also exist (globally) non-conjugate maximal orders.

Example.(Platonov & Rapinchuk, *Algebraic groups and number theory*, §8.1) Let q_1 and q_2 be integer quadratic forms given by the matrices

$$Q_1 = \begin{pmatrix} 5 & 0 \\ 0 & 11 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 1 & 0 \\ 0 & 55 \end{pmatrix}.$$

We have $q_1 \sim q_2$ over $\mathbb{Z}$ (or equivalently the lattices $\Lambda_1 = \langle 5 \rangle \perp \langle 11 \rangle$ and $\Lambda_2 = \langle 1 \rangle \perp \langle 55 \rangle$ are isometric) if and only if there exists $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z})$ such that $M^t Q_1 M = Q_2$. A direct computation shows that the existence of such a matrix M implies that the equation $5a^2 + 11c^2 = 1$ has an integer solution. Therefore, q_1 and q_2 are in different classes. But... What is its local behavior? Let's see:

by considering $M_1 = \begin{pmatrix} \frac{1}{4} & \frac{-11}{4} \\ \frac{1}{4} & \frac{5}{4} \end{pmatrix} \in GL_2(\mathbb{Z}_p)$ for $p \neq 2$ and $M_2 = \begin{pmatrix} \frac{1}{7} & \frac{-22}{7} \\ \frac{2}{7} & \frac{5}{7} \end{pmatrix} \in GL_2(\mathbb{Z}_2)$ we get

$$M_i^t Q_1 M_i = Q_2 \ (i = 1, 2).$$

Hence, q_1 and q_2 are locally equivalent (but not globally).

Despite we do not have Hasse principle in general, the local information still tell us a lot. In fact, we could say that the difference between the local and global information is finite. We state this more precisely in the better known case of quadratic forms. We will assume (for this lecture) that there exists a group $G_{\mathbb{A}}$ acting on X (with $G \subset G_{\mathbb{A}}$) such that two lattices are in the same orbit if and only if they are locally equivalent[2]. Hence, for any lattice Λ, we have $G.\Lambda \subset G_{\mathbb{A}}.\Lambda$ and we say that the **genus** $G_{\mathbb{A}}.\Lambda$ contains the **class** $G.\Lambda$. In this language, we can say that Λ satisfies the local-global principle if and only if $G.\Lambda = G_{\mathbb{A}}.\Lambda$ (class = genus). In general, a genus could contain more than one class. In the example above, we have $G.\Lambda_1 \neq G.\Lambda_2$, $G_{\mathbb{A}}.\Lambda_1 = G_{\mathbb{A}}.\Lambda_2$ and $G.\Lambda_1, G.\Lambda_2 \subset G_{\mathbb{A}}.\Lambda_1$.

Now, when we said that the difference between local and global information is finite, we meant that the number of classes in a given genus is finite[3]. This number is known as the **class number** of the object in question (quadratic lattice, skew-hermitian lattice or order of maximal rank). The quadratic and skew-hermitian cases are quite similar and we will treat the quadratic case for simplicity. From the next session on we will focus on orders in central simple algebras. General group theory tells us that the genus of a lattice Λ is in one-to-one correspondence with the set $G_{\mathbb{A}}/G_{\mathbb{A}}^{\Lambda}$, where $G_{\mathbb{A}}^{\Lambda}$ is the stabilizer of Λ in $G_{\mathbb{A}}$. It is not difficult to see that the set of classes in the genus of Λ is in one-to-one

[2]So, $G_{\mathbb{A}}$ must depend on the set of places of the number field K.

[3]See Platonov & Rapinchuk, *Algebraic groups and number theory*, §5.2.

correspondence with the set of double cosets

$$G \setminus G_{\mathbb{A}}/G_{\mathbb{A}}^{\Lambda}.$$

This set has no additional structure in general and its cardinality is in general difficult to compute. An easier problem would be to study the "abelian part", that is, the set $G_{\mathbb{A}}/GG_{\mathbb{A}}^{c}G_{\mathbb{A}}^{\Lambda}$, where $G_{\mathbb{A}}^{c}=[G_{\mathbb{A}},G_{\mathbb{A}}]$ is the commutator subgroup of $G_{\mathbb{A}}$. It turns out that[4], this group is the same as $G_{\mathbb{A}}/GG'_{\mathbb{A}}G_{\mathbb{A}}^{\Lambda}$, where $G'_{\mathbb{A}}$ is the kernel of the coboundary map[5] (spinor norm) $\Theta_{\mathbb{A}}: G_{\mathbb{A}} \to J_K/J_K^2$, where $J_K=\mathbb{A}^*$ is the idele group of K. So, we have an intermediate step[6]:

$$\begin{array}{rcl} gen(\Lambda) & = & G_{\mathbb{A}}.\Lambda \\ | & & \\ spn(\Lambda) & = & GG'_{\mathbb{A}}.\Lambda \\ || & & \text{If } G \text{ is not compact at some archimedean place.} \\ cls(\Lambda) & = & G.\Lambda \end{array}$$

The set of spinor genera in the genus of a lattice is in one-to-one correspondence with the finite abelian group

$$\Theta_{\mathbb{A}}(G_{\mathbb{A}})/\big(\theta(G)\Theta_{\mathbb{A}}(G_{\mathbb{A}}^{\Lambda})\big).$$

Morover, if we denote by $P: J_k \to J_k/J_k^2$ the canonical projection and by $H_{\mathbb{A}}(\Lambda)$ the set

$$H_{\mathbb{A}}(\Lambda)=P^{-1}\big(\Theta_{\mathbb{A}}(G_{\mathbb{A}}^{\Lambda})\big),$$

we have a group isomorphism

$$\Theta_{\mathbb{A}}(G_{\mathbb{A}})/\big(\theta(G)\Theta_{\mathbb{A}}(G_{\mathbb{A}}^{\Lambda})\big) \cong J_K/K^*H_{\mathbb{A}}(\Lambda).$$

[4]For a discution on G'/G^c in the quadratic case, see O'Meara, *Introduction to Quadratic Forms, §95, §101.*

[5]It follows by applying Galois Cohomology to the short exact sequence

$$\{1\} \longrightarrow \mu_2 \longrightarrow \widetilde{G_{\overline{K}}} \longrightarrow G_{\overline{K}} \longrightarrow \{1\},$$

where μ_2 is the group of square roots of unity, $\overline{K}$ is the algebraic closure of K and $\widetilde{G_{\overline{K}}}$ is the universal cover of G ($=O^+(V)$ in the quadratic case).

[6]In form cases, the fact that the group G is not compact at some archimedean place is equivalent to the form being isotropic, and in the CSA case it is equivalent to Eichler condition (we will see that condition in subsequent talks)

Now, the subgroup $K^*H_{\mathbb{A}}(\Lambda)$ corresponds, via class field theory, to an abelian extension Σ_Λ/K. The field Σ_Λ is called the **spinor class field** of Λ and it is useful to scrutinize almost every arithmetic aspect of spinor genera. For example, it is useful in the study of growth of the number of spinor genera under finite or infinite field extensions, in classification theory and in representation theory of our objects above[7]. Our focus here will be the representation theory of orders in central simple algebras. Let's see the general facts about the representation of lattices.

Let Λ be a lattice of maximal rank (in a quadratic or skew-hermitian space or in a CSA). We say that a lattice M (in the same space of Λ) is G-represented by Λ if there exists $g \in G$ such that $gM \subset \Lambda$. If $\mathfrak{X}$ is a set of lattices, we say that M is G-represented by $\mathfrak{X}$ if it is represented by some element of $\mathfrak{X}$. We call an adelic point $u \in G_{\mathbb{A}}$ a generator for $\Lambda|M$, if $M \subset u\Lambda$. Let $X_{\Lambda|M} \subset G_{\mathbb{A}}$ denote the set of such generators. We can assume (without lose of generality) that $M \subset \Lambda$. We have next result

Lemma. The set of spinor genera in the genus of Λ that G-represent M is in bijection with the image of $\mathfrak{H}_{\mathbb{A}}(X_{\Lambda|M})$ in the quotient $J_K/K^*H_{\mathbb{A}}(\Lambda)$, where $\mathfrak{H}_{\mathbb{A}}(X_{\Lambda|M}) = P^{-1}(\Theta_{\mathbb{A}}(X_{\Lambda|M}))$.

Let $H^-(\Lambda|M) \subset J_K$ be the subgroup generated by $K^*\mathfrak{H}_{\mathbb{A}}(X_{\Lambda|M})$. Let $H_-(\Lambda|M)$ be the maximal group H satisfying $HK^*\mathfrak{H}_{\mathbb{A}}(X_{\Lambda|M}) = K^*\mathfrak{H}_{\mathbb{A}}(X_{\Lambda|M})$. Let $\Sigma_-(\Lambda|M)$ be the class field corresponding to $H^-(\Lambda|M)$, and let $\Sigma^-(\Lambda|M)$ be the class field corresponding to $H_-(\Lambda|M)$. We call $\Sigma^-(\Lambda|M)$ the upper relative spinor class field and $\Sigma_-(\Lambda|M)$ the lower relative spinor class field. Observe that $\Sigma_-(\Lambda|M) \subset \Sigma^-(\Lambda|M)$. If $\Sigma_-(\Lambda|M) = \Sigma^-(\Lambda|M)$, we denote it by $\Sigma_{\Lambda|M}$ and call it the relative spinor class field. Hence, $\Sigma_{\Lambda|M}$ is defined if and only if $K^*\mathfrak{H}_{\mathbb{A}}(X_{\Lambda|M})$ is a group. Whenever the spinor class field is defined, the fraction of the total number of spinor genera in the genus of Λ that G-represent M is

$$\frac{1}{[\Sigma_{\Lambda|M} : K]} = \frac{|K^*\mathfrak{H}_{\mathbb{A}}(X_{\Lambda|M})/K^*H_{\mathbb{A}}(\Lambda)|}{|J_K/K^*H_{\mathbb{A}}(\Lambda)|}.$$

It is known that the relative spinor class field is defined in the quadratic and skew-hermitian cases[8]. The CSA's case will be treated in the next sessions.

[7]For the quadratic case, see J.S. Hsia, *Arithmetic of indefinite quadratic forms*, Contemporary Math., 249 (1999), 1-15

[8]For the quadratic case, see J.S. Hsia, Y.Y. Shau, F. Xu, *Representations of indefinite quadratic forms*, J. Reine angew. Math., 494 (1998), 129-140. For the skew-hermitian case, see L.E. Arenas-Carmona, *Representation fields for quaternionic skew hermitian forms*, Arch. Math. 94 (2010), 351-356.